17 Stars notes 2019/10/9 - Wed - Nuclear cross sections, S-factor

17.1 Tunneling factor

last time with r_c is defined $Z_1 Z_2 e^2/r_c = E$ the tunneling integral is

$$I = \int k dr = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2} \int_{r_n}^{r_c} \left[\frac{r_c}{r} - 1\right]^{1/2} dr = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2} r_c \int_{r_n/r_c}^{1} \left[\frac{1}{x} - 1\right]^{1/2} dx$$

we will end up taking $r_n \to 0$ which is a good approximation for this integral. This integral ends up

$$I = \frac{\pi}{2} \alpha Z_1 Z_2 \left(\frac{2m_r c^2}{E}\right)^{1/2}$$

where α is the fine structure constant. The probability we want is

$$P(0)/P(\infty) = \exp\left(-\pi \alpha Z_1 Z_2 \left(\frac{2m_r c^2}{E}\right)^{1/2}\right)$$

Called the Gamov tunnelling factor. There is an energy scale that we can write as the Gamov energy

$$E_G = (\pi \alpha Z_1 Z_2)^2 (2m_r c^2)$$

so that

$$P(0) \propto \exp\left(-(E_G/E)^{1/2}\right)$$

Looking at some numbers: for $p+p$ we have $E_G = 494$ keV and for $p+^{12}$C have $E_G = 33$ MeV. This gives for $p+p$,

$$P(0) = \exp\left(-\left(\frac{494}{1}\right)^{1/2}\right) = 2 \times 10^{-10}$$

this is how likely you are to see the nuclear force, but even if you do because we want a weak interaction it is suppresed by another large factor (10^{-25}).

for $E = \frac{1}{2} m_r v^2$ (v = relative velocity) can rewrite

$$P(0) = \exp\left(-2\pi \frac{e^2 Z_1 Z_2}{\hbar v}\right)$$

There is in some sense a maximum coulomb energy

$$E_{coul} = \frac{Z_1 Z_2 e^2}{\lambda} = \frac{Z_1 Z_2 e^2 m_r v}{h}$$

then

$$P(0) \text{ scales like } \exp\left(-\frac{E_{coul}(\lambda)}{\frac{1}{2} m_r v^2}\right)$$

Now want to convolute this probability with the thermal distribution to see what energy particles are important.

17.2 Nuclear cross-sections

We've only calculated the overlap. There's reason to expect a resonance which helps the cross section. For now consider non-resonant reactions. The only length scale in the problem is

$$\lambda = \frac{h}{p} \simeq 10^{-10}\text{cm at keV}$$

this is much greater than $R_{nuc} = 1.3$ fm $A^{1/3}$. The reaction physics depends only on the COM energy

$$E_{COM} = \frac{1}{2} m_r |\vec{v}_1 - \vec{v}_2|^2$$

For s-wave interactions (no angular momentum in COM) the cross section, which is what one measures with an accelerator, looks like

$$\sigma(E) = (4\pi\lambda^2)(\text{dimensionless stuff}) \exp(-(E_G/E)^{1/2})$$

the way this is defined (consistent with textbook)

$$4\pi\lambda^2 = 4\pi\frac{1}{k^2} = \frac{2\pi\hbar^2}{m_r E_c} = 2000 \text{ barns} \left(\frac{keV}{E_{com}}\right)$$

where 1 barn $= 10^{-24}$ cm^2 and $E_{com} = \hbar^2 k^2/2m$.

Then the conventional way to write the cross section is

$$\sigma(E) = \frac{1}{E} S(E) e^{-(E_G/E)^{1/2}}$$

Where $S(E)$ is called the "S-factor", and is what is usually obtained in the lab and extrapolated (more on this next time). If the dimensionless stuff was $= 1$ then

$$S(E) = 2000 \text{ barns keV}$$

This is a VERY large cross section, rarely are they this big. Now the experimentalists must measure the dimensionless stuff for the reaction we want, that is get $S(E)$. To do this experimentalists must measure $\sigma(E)$ and then *extrapolate* to low energies.

The cross section $\sigma(E)$ is what is appropriate for an experiment. i.e. a beam of well-determined energy pointed at a target. By contrast, stars will have a broad thermal energy distribution.

An example of a measured cross-section for $p+^{12}$C from Clayton chapter 4:

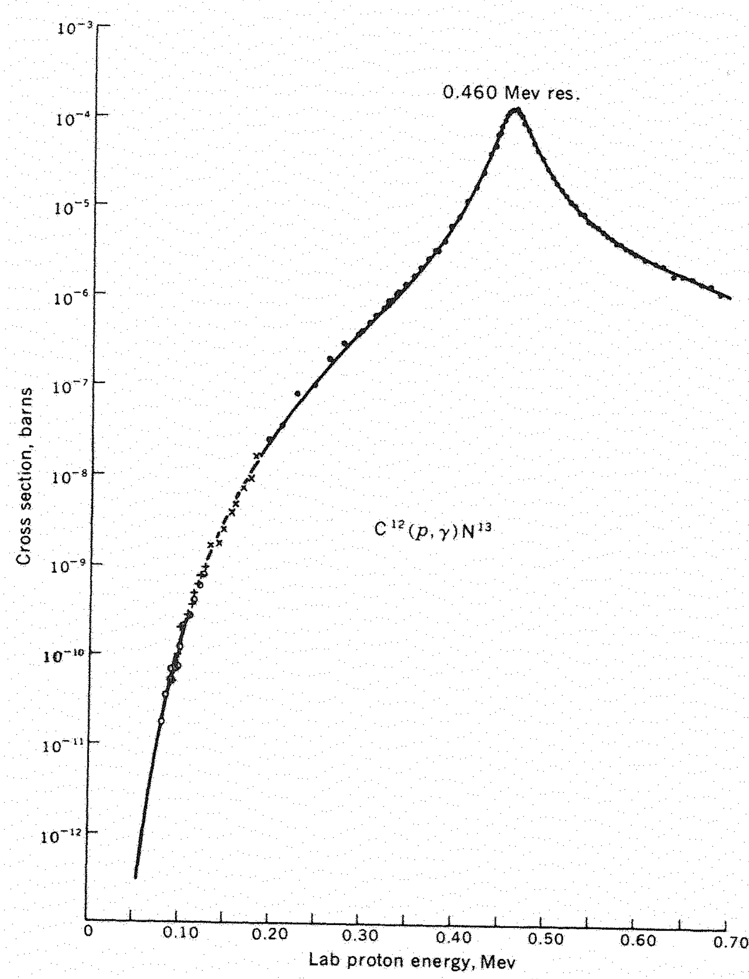

Fig. 4-4 The measured cross section for the reaction $C^{12}(p,\gamma)N^{13}$ as a function of laboratory proton energy. A four-parameter theoretical curve has been fitted to the experimental points. An extrapolation to $E_p = 0.025$ Mev, which is an interesting energy for this reaction in astrophysics, appears treacherous. (*Courtesy of W. A. Fowler and J. L. Vogl.*)

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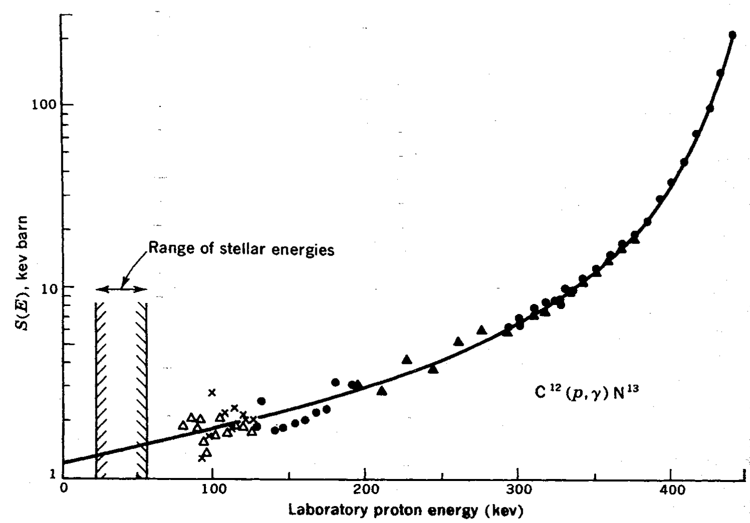

Fig. 4-5 **The cross-section factor $S(E)$ for the radiative capture of protons by C^{12}. Th differing types of data points represent five different experiments performed at different ti and laboratories by the workers indicated. Detailed references and discussion may be found i D. F. Hebbard and J. L. Vogl, *Nucl. Phys.*, 21:652 (1960). This curve is more readily extrapo- lated than the one in Fig. 4-4.**

Another example from $^{12}C+^{16}O$, which is very important for thermonuclear supernovae (Martinez-Rodriguez et al. 2017ApJ...843...35M) is the following cross section from Jiang et al. (2007).

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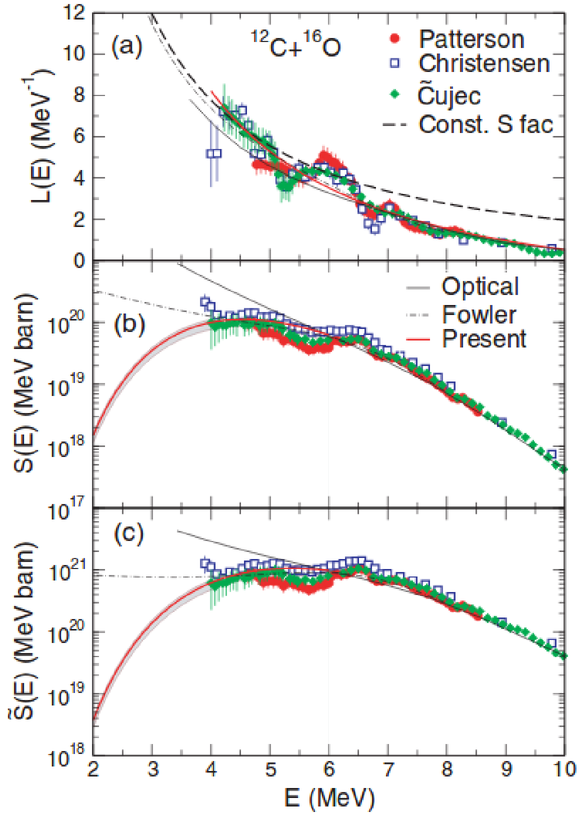
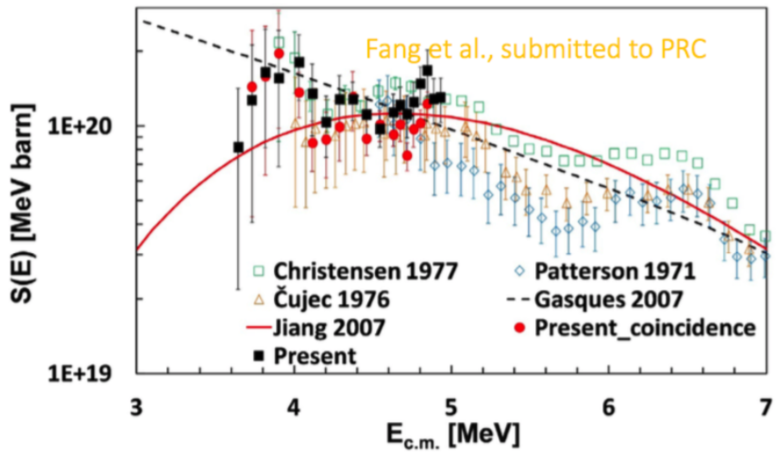


FIG. 6. (Color) Plot of (a) $L(E)$, (b) $S(E)$ and (c) $\tilde{S}(E)$ for the fusion reaction ^{12}C+^{16}O with data taken from Refs. [10–12]. The derivatives were obtained by least-squares fits to eight, five, and five cross section points for Patterson's, Christensen's, and Čujec's experiments, respectively. The black long-dashed line corresponds to a constant S factor, while the black solid line is the result of an optical model calculation. The black dot-dashed curves correspond to the fit obtained by Fowler *et al*. [2]. The red solid lines are the results of least-squares fits to the data using Eqs. (10) and (11). See text for details.

And from pre-release data:



We need to figure out what energy range is important for stars.

17.3 (thermal) Nuclear reactions for star

Need to go from cross section $\sigma(E)$, which depends on incoming particle energy, to rate $\epsilon(T)$, which depends on gas temperature.

Now we want to go about convolving with the thermal distribution of the protons. Usually we write down time between collisions

$$t \simeq \frac{1}{\sigma n v} \quad \text{or a rate} \quad r = \sigma n v$$

We have $\sigma(E)$, dependent on energy of incoming particle, but we really want the thermal average

$$\langle \sigma v \rangle = \int_0^\infty v_r \sigma(v_r) P(v_r) dv_r$$

where $P(v_r)dv_r$ = probability that two particles have a relative velocity between v_r and $v_r + dv_r$.

The energy generation rate will then be

$$\epsilon_{nuc} = \frac{n_1 n_2 \langle \sigma v \rangle E_{\text{nuc}}}{\rho}$$

where E_{nuc} is the energy released per reaction.

For an ideal gas obeying maxwell-Boltzmann:

$$P(\vec{v})d^3\vec{v} = \left[\frac{m_r}{2\pi kT}\right]^{3/2} \exp\left(-\frac{m_r v_r^2}{2kT}\right) d^3\vec{v}$$

now just need to put this in and do the integral. The integral becomes

$$\langle \sigma v \rangle = \left[\frac{m_r}{2\pi kT}\right]^{3/2} \int_0^\infty \exp\left(-\frac{m_r v_r^2}{2kT}\right) 4\pi v_r^2 dv_r \cdot v_r \sigma(v_r)$$

Will put in $\sigma(E)$ and discuss this integral next time. This will tell us the energy of typical reacting particles.