16 Stars notes 2019/10/7 - Mon - Hydrogen fusion

16.1 Transition to nuclear fusion

What do we know?

1. hydrostatic balance and ideal gas $kT \simeq GMm_p/R$
2. Luminosity set by the rate of heat transport.
3. Stars can live on gravitational contraction for about

$$t_{KH} \sim \frac{GM^2/R}{L} \simeq 10^7 years$$

The *Radius* of main sequence stars will be fixed by condition that

$$L = \int \epsilon_{nuc} dm$$

The big bang makes by mass 75% H and about 25% He, and some ^{7}Li, ^{3}He, D.

We want to take advantage of the $\sim$ 7 MeV binding energy/nucleon to power the sun and other stars for $\sim$ Gyrs.

16.1.1 The *pp* chain

Starting from pure Hydrogen. First we need some neutrons

$$p + p \rightarrow d + e^+ + \nu_e$$

this is the first step, also the slowest step since it is a weak interaction.

$$p + d \rightarrow^3 He + \gamma$$

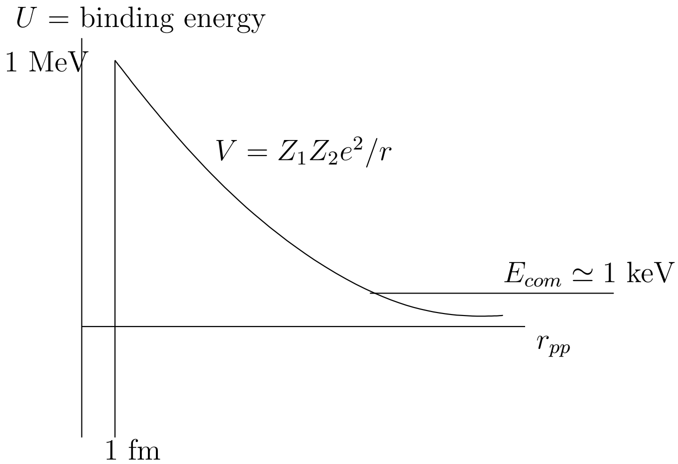
usually closed by

$$^3He +^3 He \rightarrow^4 He + 2p$$

This is called the *pp* cycle. The last two steps are strong interactions and so are fast. The last has a large coulomb barrier.

16.2 Coulomb Tunneling in nuclear reactions

The nuclear force is short ranged distances $\simeq$ fm so we need to find the overlap of the wavefunction of two charged particles on this length scale.



There is about a 1 MeV coulomb energy barrier to getting into the nucleus, wheras the energy of the protons is more like 1 keV.

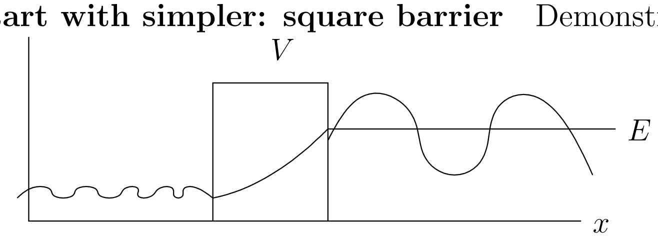
What is the de Broglie λ of the protons?

$$E_{th} = 1keV \Rightarrow \lambda_{prot} = \frac{h}{p} = 10^{-10} \text{ cm}$$

Even the QM "size" of the proton is $\gg r_{nuc} \simeq 10^{-13}$ cm. Coulomb tunneling for scattering states $\Rightarrow$ Two incident plane waves and ask what is the overlap probability at $r = 0$.

We'll do just the one dimensional problem in WKB in a moment.

Start with simpler: square barrier Demonstrate concept of tunneling



For tunneling through a square barrier of height V we solve schrodinger's equation:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(x) = E\psi(x)$$

the mass here in the actual problem will be the reduced mass

$$m = m_r = \frac{m_1 m_2}{m_1 + m_2}$$

Away from the barrier the wave function looks like $\psi(x) = e^{ikx}$, so that

$$\frac{\hbar^2 k^2}{2m} = E$$

that relates E and the wavenumber of the particle $k = 2\pi/\lambda$

Inside the barrier we have

$$\frac{-\hbar^2}{2m_r}\frac{d}{dx^2}\psi = (E - V)\psi$$

with $V > E$ so that the solution is $\psi \sim e^{\pm kx}$. Then the probability for penetration of the barrier comes from the square of the wave function on the left of the barrier:

$$|\psi(0)|^2 = [e^{-k_{in}\Delta x}]^2 |\psi_\infty|^2$$

is the probability at the orgin for a particle coming through a barrier from infinity.

Now with actual potential and WKB approximation For our case, in the WKB approximation, where it is assumed $1/k \ll$ length scale of variations in $V(r)$, assume $\psi(r) \propto e^{kx}$, but now $k(r)$ can vary in space. By only keeping first-order terms, the Schrodinger equation is then

$$-\frac{\hbar^2[k(r)]^2}{2m_r} = -\frac{Z_1 Z_2 e^2}{r} + E$$

i.e. instead of solving the "real" ODE, we just assume the wave function is everywhere like a plane wave (e^{ikx}) and solve for $k(r)$. Like in barrier problem we want

$$P \sim |e^{-k_{in}\Delta z}|^2 \Rightarrow \text{in WKB this becomes} \Rightarrow \exp\left(-2\int k(r)dr\right) .$$

The wave number is

$$k(r) = \left(\frac{2m_r}{\hbar^2}\right)^{1/2} \left[\frac{Z_1 Z_2 e^2}{r} - E\right]^{1/2} = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2} \left[\frac{r_c}{r} - 1\right]^{1/2}$$

where r_c is defined $Z_1 Z_2 e^2 / r_c = E$ then the integral is

$$I = \int k dr = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2} \int_{r_n}^{r_c} \left[\frac{r_c}{r} - 1\right]^{1/2} dr = \left(\frac{2m_r E}{\hbar^2}\right)^{1/2} r_c \int_{r_n/r_c}^{1} \left[\frac{1}{x} - 1\right]^{1/2} dx$$

we will end up taking $r_n \to 0$ which is a good approximation for this integral. This integral ends up

$$I = \frac{\pi}{2}\alpha Z_1 Z_2 \left(\frac{2m_r c^2}{E}\right)^{1/2}$$

where α is the fine structure constant. The probability we want is

$$P(0) \propto \exp\left(-\pi\alpha Z_1 Z_2 \left(\frac{2m_r c^2}{E}\right)^{1/2}\right)$$

Called the Gamov tunnelling factor.