15 Stars notes 2019/10/2 - Wed - Protostar contraction time

15.0.1 Entropy/energy Equation

We've written down hydrostatic balance and heat transfer, now we need the energy equation.

$$T\frac{ds}{dt} = \frac{dQ}{dt} = \text{energy lost or gained by a fluid element}$$

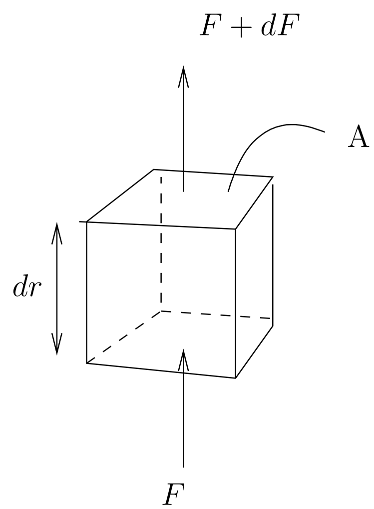
Several ways for a fluid element to change energy

1. Heat gained by nuclear reaction ϵ units erg/gr s

2. Heat gained (or lost) by transport

$$-\frac{1}{\rho}\vec{\nabla}\cdot\vec{F}$$

which has units erg/gr s, where F is the heat flux. easy to derive: flux in F, flux leaving $F + dF$.



so then the energy gained is

$$\frac{\text{gain}}{\text{mass}} = \frac{FA - (F + dF)A}{\rho A dr}$$

and the mass of the element is $\rho A dr$ so we find $-\frac{dF}{dR}\frac{1}{\rho}$. Another way, is that the flux is

$$\int \vec{F}\cdot d\vec{A} = \int \vec{\nabla}\cdot\vec{F}dV$$

so that we get the heating above.

Now we have

$$T\frac{ds}{dt} = \epsilon_{nuc} - \frac{\vec{\nabla}\cdot\vec{F}}{\rho}\ .$$

For spherical symmetry, rewrite $F = F_r\vec{\hat{r}}$ and then

$$\frac{\nabla\cdot F}{\rho} = \frac{1}{\rho}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) = \frac{1}{\rho 4\pi r^2}\frac{\partial}{\partial r}(L_r)$$

which is

$$\frac{\partial L_r}{\partial m}$$

so that the equation is

$$T\frac{ds}{dt} = \epsilon - \frac{dL_r}{dm}$$

where

$$m(r) = \int_0^r \rho 4\pi r^2 dr$$

15.0.2 Rephrase LHS in terms of change in state in fluid

Rewrite entropy in terms of more useful fluid variables like E, T, P, ρ,

$$T\frac{ds}{dt} = \frac{dE}{dt} - \frac{P}{\rho^2}\frac{d\rho}{dt}$$

where E is the internal energy per gram. Ideal gas

$$E = \frac{3}{2}\frac{kT}{\mu m_p} = \frac{3}{2}\frac{P}{\rho}$$

so that

$$\frac{dE}{dt} = \frac{3}{2}\frac{1}{\rho}\frac{dP}{dt} - \frac{3}{2}\frac{P}{\rho^2}\frac{d\rho}{dt} = \frac{3}{2}\frac{P}{\rho}\frac{d\ln P}{dt} - \frac{3}{2}\frac{P}{\rho}\frac{d\ln\rho}{dt}$$

and so we can rewrite Tds in terms of P and ρ,

$$T\frac{ds}{dt} = \frac{P}{\rho}\left[\frac{3}{2}\frac{d\ln P}{dt} - \frac{5}{2}\frac{d\ln\rho}{dt}\right] = \frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\ln(P/\rho^{5/3})$$

so that if $P \propto \rho^{5/3}$, $T\frac{ds}{dt} = 0$.

To find time evolution of P, ρ and thus $R(t)$, i.e. contraction, we will equate this to $\epsilon - \frac{dL}{dm}$.

15.0.3 Application to pre-main-sequence contraction

For a fully convective , pre-main sequence star the entropy is the same everywhere, but changing with time. i.e. $P/\rho^{5/3}$ is the same at all locations in star, but varies in time. We can then relate this to how the radius changes in time.

How does $P/\rho^{5/3}$ scale with the radius of the star? Star is in hydrostatic balance so that

$$P_c \sim \frac{GM^2}{R^4}, \quad \rho_c \sim \frac{M}{R^3}$$

we can eliminate R from these in favor of ρ_c by writing

$$R \sim \left(\frac{M}{\rho_c}\right)^{1/3}$$

which gives

$$P_c \sim \frac{GM^2}{(M^{1/3}\rho_c^{1/3})^4} \propto \rho_c^{4/3}$$

where we are ignoring the M dependence since this that is constant during contraction. This can be used to evaluate

$$\frac{P_c}{\rho_c^{5/3}} \propto \rho_c^{-1/3} = \frac{P}{\rho^{5/3}} \text{ for whole star}$$

This relates a quantity everywere in the whole star $(P/\rho^{5/3})$ to the overall radius that appears in ρ_c. So the entropy equation becomes

$$T\frac{ds}{dt} = \frac{3}{2}\frac{P}{\rho}\frac{d}{dt}\left(\ln\frac{P}{\rho^{5/3}}\right) \sim -\frac{1}{2}\frac{P}{\rho}\frac{d}{dt}\ln\rho_c$$

For the pre main sequence as it contracts $\underline{\rho_c \text{ increases}}$ so that $\underline{T\frac{ds}{dt} < 0}$ so that the entropy decreases as the star contracts. radiated in the photons.

Want to get an equation for the radius $R(t)$. Put $\rho \sim M/R^3$ in to get R in terms of Tds/dt,

$$T\frac{ds}{dt} = \epsilon - \frac{dL}{dm} \quad \text{becomes} \quad \frac{3}{2}\frac{P}{\rho}\frac{1}{R}\frac{dR}{dt} = -\frac{dL_r}{dm_r}$$

the luminosity is

$$L = -\frac{3}{2}\frac{1}{R}\frac{dR}{dt}\int\frac{P}{\rho}dm_r$$

and then

$$L = \frac{3}{2}\frac{1}{R}|\frac{dR}{dt}|\int_0^R P4\pi r^2 dr$$

Since the star is in Hydrostatic balance we can use the virial theorem to relate the integral $\int P4\pi r^2 dr$ to $(..)GM^2/R$. For a star with $P(R) \propto \rho(r)^{5/3}$ then (for this polytrope)

$$\int PdV = \frac{2}{7}\frac{GM^2}{R}$$