12 Stars notes 2019/09/25 - Wed - convection

We will start stellar evolution with protostars - which as we have seen in MESA are fully convective. So we need to understand convection first!

12.1 Condition for Convection

So we found that, for stability

$$\frac{d\ln\rho_{\rm bub}}{dr} > \frac{d\ln\rho}{dr}$$

or, using the relation we found for a bubble rising adiabatically in pressure equilibrium,

$$\frac{d\ln\rho_{\rm bub}}{dr} = \frac{1}{\gamma}\frac{d\ln P}{dr}$$

where $\gamma = 5/3$ for monatomic ideal gas, this becomes

$$\frac{1}{\gamma}\frac{d\ln P}{dr} > \frac{d\ln\rho}{dr}$$

It is most conventient to have a relation between T and P. To do this, we have to presume some equation of state. If we presume an ideal gas, then $P = \rho kT/\mu m_p$, or equivalently $\rho = \mu m_p P/kT$. Now eliminate ρ in favor of T and P using this EOS to relate derivatives:

$$d\ln\rho = d\ln P - d\ln T + d\ln\mu$$

for constant μ (i.e. uniform composition), $d\ln\mu = 0$. Putting this into our inequality gives

$$\left(\frac{1}{\gamma} - 1\right)\frac{d\ln P}{dr} > -\frac{d\ln T}{dr}$$

for stability. Since both things on the left are negative, the LHS is a positive quantity, also the RHS is positive since T increases into the star. Putting these in explicitly gives

$$\left|\frac{d\ln T}{dr}\right| < (1 - \frac{1}{\gamma})\left|\frac{d\ln P}{dr}\right|$$

or

$$\boxed{\frac{d\ln T}{d\ln P}|_* < 1 - \frac{1}{\gamma}}$$

to be stable. for and ideal gas $1 - 1/\gamma = 2/5$. More generally this is stated

$$\nabla < \nabla_{\rm ad}$$

where $\nabla = d\ln T/d\ln P$ in the stellar profile and $\nabla_{\rm ad} = \partial\ln T/\partial\ln P$ evaluated at constant entropy, a local EOS property.

Let's check a model. For the Eddington standard model $P \propto T^4$ so that

$$\frac{d\ln T}{d\ln P}|_{Edd} = \frac{1}{4}$$

so that model is stable, no convection.

Hansen, Kawaler, & Trimble figure 5.2 for sun.

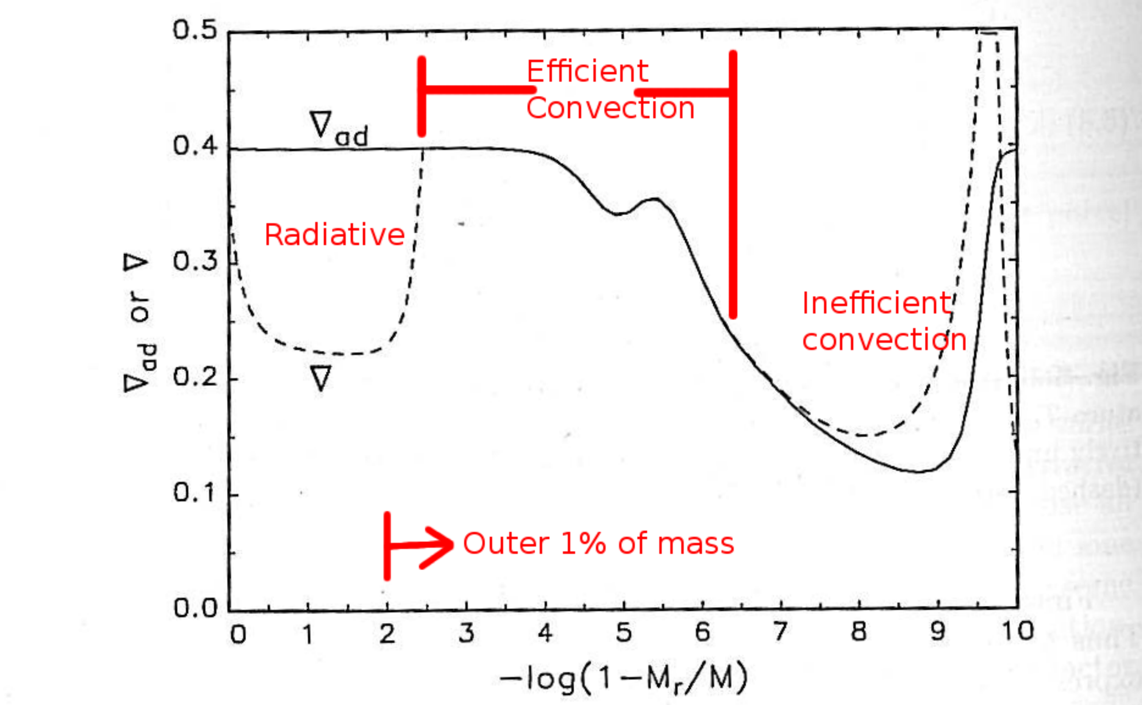
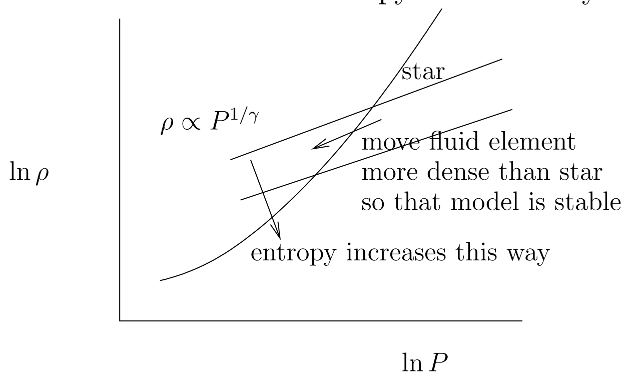


Fig. 5.2. Shown are ∇_{ad} (solid line) and ∇ (dashed line) versus $-\log(1-\mathcal{M}_r/\mathcal{M})$ for a model ZAMS sun. See also Fig. 3.11.

12.2 Entropy profile

Let's redo this in a more entropy centered way.



The adiabat is for a particular entropy of the fluid element. What is the T-ρ relation for an adiabat?

$$TdS = dE + pdV$$

for an adiabat

$$0 = \frac{3}{2}Nk_B dT + \frac{N}{V}kTdV$$

or

$$\frac{3}{2}\frac{dT}{T} = -\frac{dV}{V}$$

so then for an adiabat $T \propto V^{-2/3}$ or $T \propto \rho^{2/3}$ for a adiabat.

But now if I want to increase entropy. At fixed ρ I want to increase S. to do so must increase T, thus the pressure increases at fixed ρ as S increses. So a curve to the right on the above plot has higher entropy. So we have found that entropy decreasing inwards is stable, and entropy increasing inward is unstable. (note that a steeper temperature gradient implies a shallower density gradient, for the same pressure.)

12.3 Action of convection

Back to considering a fluid element moving from 1 to 2, a distance Δr. so for the bubble

$$\rho_{2,bub} = \rho_1 \left(\frac{p_2}{p_1}\right)^{1/\gamma}$$

want an idea how the bubble accelerates as it moves up. Also,

$$\rho_{2,*} = \rho_1 + \Delta r \frac{d\rho}{dr}|_*$$

density contrast between the bubble and the star,

$$\Delta\rho = \rho_{2,*} - \rho_{2,b}$$

which is positive for the unstable case. After some work like above,

$$\Delta\rho = \rho\Delta r \left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right]$$

where we have used $\frac{dP}{dr} = -\rho g$. The acceleration on the fluid element due to buoyancy is

$$a = \frac{\Delta\rho}{\rho}g = g\Delta r \left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right]$$

The thing on the right without the Δr is a number set by the model that has units of $1/sec^2$. It is convenient to define the Brunt Vaisala frequency

$$N_{BV}^2 \equiv -g\left[\frac{d\ln\rho}{dr} + \frac{\rho g}{P\gamma}\right] \quad \text{so that} \quad a = -N_{BV}^2\Delta r$$

Note that if the term in [..] in the equation of motion is negative we have a simple harmonic oscillator. Let's consider this case first.

When the star has a stable gradient $N^2 > 0$ then $a = -\Delta r N^2$ or $\ddot{x} = -xN^2$ which gives simple harmonic motion at frequency N. This is very important for stellar oscillations. This mechanism gives gravity waves in the earths atmosphere.

So, what is N^2 approximately? The two terms are comparable, so we'll just use $\frac{d\ln\rho}{dr} \approx 1/H$ so then

$$N^2 = \frac{g}{H}$$

which since $H = kT/m_p g$ is the scale height, $N^2 = m_p g^2/kT$ or

$$N \simeq g\left(\frac{m_p}{kT}\right)^{1/2} \approx \frac{g}{v_{th}} \approx \frac{g}{c_s}$$

What is N near the center,

$$N^2 \sim \frac{g}{R} \sim \frac{GM}{R^3}$$

so that in this case $N \sim 1/t_{dyn}$.

12.4 Motion and Profile in Unstable Case

Will consider in 2 steps:
1. What speeds are possible?
2. What temperature gradient is required?

What speeds are possible? assuming unstable, then the equation of motion is

$$\ddot{x} = x\frac{1}{\tau^2}$$

so that $x = x_0 e^{t/\tau}$ and the velocity of the fluid element is

$$v = \frac{x_0}{\tau}e^{t/\tau}$$

so that if moved $\ell = x_0 e^{t/\tau}$ we have

$$v = \frac{\ell}{\tau}$$

velecity increases linearly with distance moved. Putting stuff back in

$$v = \ell\sqrt{g\left(\frac{d\ln\rho}{dr} + \frac{\rho g}{\gamma P}\right)}$$

want to write this in terms of the density contrast at position ℓ. At position ℓ,

$$\frac{\Delta\rho}{\rho}|_{at\ell} = \ell\left(\frac{d\ln\rho}{dr} + \frac{\rho g}{\gamma P}\right)$$

identifying that in the v above,

$$v = (g\ell)^{1/2} \left(\frac{\Delta\rho}{\rho}|_\ell \right)^{1/2}$$

Just put in $\ell = H = kT/m_p g$ to find the velocity upon moving a scale height

$$v = \left(\frac{gkT}{m_p g} \right)^{1/2} \left(\frac{\Delta\rho}{\rho}|_\ell \right)^{1/2} \simeq c_s \left(\frac{\Delta\rho}{\rho}|_{\ell=H} \right)^{1/2}$$

so then if you've moved a scale height you'll be moving at approximately the sound speed (if $\Delta\rho/\rho \sim 1$). This really results from the definition of the scale height, moving this far gives you about the thermal energy. $\Delta\rho/\rho = 1$ is incredibly fast. Since it is so efficient $\Delta\rho/\rho$ small is plenty to move heat out. This being small means that the whole convective region is on an adiabat and thus is at constant entropy.