9 Stars notes 2019/09/11 - Wed - polytropes, Eddington model

9.1 Polytropes

Slightly better model of internal structure Another way to get to a similar endpoint as previous finite-differencing. A bit more quantitative, and will be used to construct the Eddington standard model.

Already shown that ideal gas pressure is dominant in most cases,

$$P(r) = \frac{\rho(r) k_B T(r)}{\mu m_p}$$

If we can relate $T(r)$ to $\rho(r)$ somehow, then $P(r)$ is a function of $\rho(r)$ only. Then the structure can be solved for self-consistently. Can see this from the following, recall we have two equations

$$\frac{dP}{dr} = -\frac{\rho(r) G m(r)}{r^2}$$

and

$$m(r) = \int_0^r \rho(r) 4\pi r^2 dr$$

when $P = K\rho^\alpha$ then the structure can be solved for with a given ρ_c or P_c. Done by using the polytropic EOS (the P-ρ relation) to convert P to ρ in the above equations and then integrating both from the center. Generally needs to be done numerically, but it can be done. The results are called Lane-Embden functions. These are called polytropes!

Ofter n is used for the index, but it is conventionally not the same as the α in $P = K\rho^\alpha$ above, instead $\alpha = 1 + 1/n$, but you should check the context.

Once the integration is performed, one is able to map (K, α) and ρ_c to full profiles $\rho(r)$, $P(r)$ and also total mass M and radius R. This relation means that one only needs to specify 3 parameters and that will give all the others. Also note that K can be obtained from ρ_c, P_c and α if all of them are given. So it is possible to get a structure if any one of the following combinations of three are given: (α,ρ_c, P_c), (α, K, P_c), (α, K, M), (α, M, R). Note the latter ones are the hardest to use, because one basically must find the K and ρ_c that gives that M and R when integrated.

9.2 Eddington's standard model - why stars are nearly polytropes

Now we will derive a relation between P and T that allows us to use the polytropic EOS and therefore polytropic structure.

Uses radiative transport to relate ρ to T. The flux is

$$F = -\frac{1}{3} c \frac{1}{n_e \sigma_\gamma} \frac{d}{dr} a T^4$$

assuming all that matters is Thompson scattering σ_γ is the Thompson cross section for $e - \gamma$ scattering. But rather than writing cross sections we like to define the opacity

$$\kappa = \frac{\sigma}{m_p} \approx \text{cm}^2/\text{gr}$$

Then the flux is

$$F = -\frac{1}{3}\frac{c}{\rho}\frac{1}{\kappa}\frac{d}{dr}aT^4(r)$$

in a real star κ could be complicated, but ours here is simple.

This gives that

$$L_r = -4\pi r^2 \frac{1}{3}\frac{c}{\rho\kappa}\frac{d(aT^4)}{dr}$$

so then L_r is the luminosity at radius r. Can write this as the gradient of the radiation pressure,

$$\boxed{L_r = -4\pi r^2 \frac{c}{\rho\kappa}\frac{d}{dr}P_{rad}}$$

where $P_{rad} = \frac{1}{3}aT^4$ now remember that for hydrostatic balance

$$\frac{1}{\rho(r)}\frac{dP}{dr} = -\frac{Gm(r)}{r^2}$$

Since $\rho(r)dr$ appears in both of these equations, we define the column by $dy = -\rho(r)dr$ so then our equations become

$$L_r = 4\pi r^2 \frac{c}{\kappa}\frac{d}{dy}P_{rad}$$

or

$$\boxed{\frac{dP_{rad}}{dy} = \frac{L(r)}{4\pi r^2}\frac{\kappa}{c}}$$

and

$$\boxed{\frac{dP}{dy} = \frac{Gm(r)}{r^2}}$$

So now we have two equations for stellar structure.

To eliminate radius, and since P_{rad} is really just an alias for the temperature, we take the ratio of these two equations.

$$\frac{dP}{dP_{rad}} = \frac{4\pi Gm(r)c}{\kappa L(r)}$$

which is some dimensionless number. writing this a bit different:

$$= \frac{4\pi Gc}{\kappa}\frac{M}{L}\left[\frac{m(r)}{M}\right]\left[\frac{L}{L(r)}\right] = \frac{L_{Edd}}{L}\left[\frac{m(r)}{M}\right]\left[\frac{L}{L(r)}\right]$$

the constant out front is unitless, so we identify the Eddington luminosity to be

$$L_{Edd} = \frac{4\pi GcM}{\kappa} = \frac{4\pi GcMm_p}{\sigma_{Th}}$$

in units this is

$$L_{Edd} = 1.2 \times 10^{38} \text{ ergs/s} \frac{M}{M_\odot} = 3.13 \times 10^4 L_\odot \frac{M}{M_\odot}$$

We talked about last time that for stars, generally

$$L = L_\odot \left(\frac{M}{M_\odot}\right)^3$$

so we see that

$$\frac{L}{L_{Edd}} \approx 3 \times 10^{-5} \left(\frac{M}{M_\odot}\right)^2$$

This is a small ratio for most stars. But at $M \geq 100 M_\odot$, $L/L_{Edd} \sim 1$. (leads to mass loss)

"The trick" is to take the combination appearing here

$$\eta(r) = \frac{m(r)}{M}\frac{L}{L(r)}$$

and assume it is some number independent of r. *This is the definitive assumption of the* **Eddington Standard Model**. So then

$$\frac{dP}{dP_{rad}} = \frac{L_{Edd}}{L}\eta$$

which gives

$$\int_R^r dP = \frac{L_{Edd}}{L}\eta \int_R^r dP_{rad}$$

so then

$$P(r) = \frac{L_{Edd}}{L}\eta P_{rad}(r)$$

This relates P and T and therefore can give us a polytrope.

As an example, to see this quickly, take the gas-pressure-dominated limit, when $L \ll L_{Edd}$, $P(r) = P_{gas}$ and then

$$\frac{\rho(r) k_B T(r)}{\mu m_p} = \frac{L_{Edd}}{L}\eta \frac{1}{3} a T^4(r)$$

so we have the relation we wanted $\rho(r) \propto T^3(r)$. Putting stuff in

$$P = P_g \propto \rho(r) T(r) \propto \rho^{4/3}(r)$$

This then allows for an integration to get the total structure.

This was for the gas pressure dominating the total pressure. We really need to allow for $P_{rad} \sim P_{tot}$. Note that the ratio of P_g to P_r is a constant with radius. Now use $P_{gas}(r) = \beta P_{tot}$ so that

$$P_{rad} = \frac{1-\beta}{\beta} P_{gas}$$

(this is how the actual Eddington Standard Model is typically phrased) so going back and doing the same thing as before but keeping β,

$$T(r) = \left[\frac{3k_B}{a\mu m_p} \frac{1-\beta}{\beta} \right]^{1/3} \rho^{1/3}(r)$$

then

$$P(r) = \left[\left(\frac{k_B}{\mu m_p} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right]^{1/3} \rho^{4/3}(r)$$

this thing out front is constant. Now you can take this back and construct an $\alpha = 4/3$ polytrope by integrating and you get

$$\frac{1-\beta}{\beta^4} = 3 \times 10^{-3} \mu^4 \left(\frac{M}{M_\odot} \right)^2$$

checking some values

$\mu^2 M/M_\odot$	β
1	0.997
2	0.9885
5	0.9412
10	0.8463
50	0.5

for a normal star $\mu = 0.6$ so then can do

$M/M_\odot$	P_r/P_g
2.8	3×10^{-3}
14	6.2×10^{-2}
138	1

As the P_r kicks in, the luminosity starts to go like the mass rather than the mass cubed. Massive stars asymtote to the eddington limit.

In homework 2, will compare mesa models to polytropes.