8 Stars notes 2019/9/9 - Mon - heat transport, scalings summary

8.0.1 What about photons?

Will first show when photons are "better" at carrying heat. Then come back to solar luminosity.

For radiation we would still write down a mean free path:

$$\ell_\gamma = \frac{1}{n_e \sigma_{Th}}$$

Assume that e-photon is Thomson scattering, $\sigma_{\gamma-e} = \sigma_{Th}$. $E = aT^4$ then

$$F_x = -\frac{1}{3} v\ell \frac{dE}{dx} = -\frac{1}{3} c \frac{1}{n_e \sigma_{\gamma-e}} \frac{d}{dx}(aT^4) = -K_r \frac{dT}{dx}$$

so that

$$K_r = \frac{1}{3} c \frac{4aT^3}{n_e \sigma_{\gamma-e}}$$

want to compare this with the electron version from above,

$$K_{gas} = v \frac{1}{\sigma_{e-i}} k_B$$

We know photons move faster, we need to check the mean free path and the energy gradient.

So compare γs to electrons for heat transport by comparing heat diffusion coefficients K,

$$\frac{K_r}{K_{gas}} = \frac{\frac{4}{3} aT^3 \frac{1}{n_e \sigma_{\gamma-e}} c}{k_B \frac{1}{\sigma_{e-i}} v} = \frac{\frac{1}{3} aT^4 \frac{1}{\sigma_{\gamma-e}} c}{\frac{1}{4} n_e k_B T \frac{1}{\sigma_{e-i}} v} \approx \frac{P_{rad}}{P_{gas}} \frac{\sigma_{e-i}}{\sigma_{e-\gamma}} \frac{c}{v_e}$$

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now we want to make each of these comparisons. first the relative cross sections

$$\sigma_{e-i} = \frac{e^4}{(kT)^2} \quad \sigma_{\gamma-e} = \frac{8\pi}{3} \frac{e^4}{(m_e c^2)^2}$$

so we see that for $kT < m_e c^2$ the e-i is bigger and so

$$\frac{K_r}{K_g} \simeq \frac{P_{rad}}{P_{gas}} \left(\frac{m_e c^2}{kT} \right)^{5/2}$$

the second ratio is very large, we are then interested in the first one. Last time we showed that if a star is supported by relativistic particles then $E_{tot} \to 0$, now we want to know what the detailed ratio is.

Let's assume radiation pressure is small, then for a constant density star held up by gas pressure:

$$kT \approx \frac{1}{10}\frac{GMm_p}{R}$$

for the gas pressure we'll use virial stuff

$$P_{gas} = -\frac{1}{3}\frac{E_{GR}}{V} = \frac{-1}{3}\frac{1}{4\pi R^3/3}\frac{3}{5}\frac{GM^2}{R} = \frac{3}{20\pi}\frac{GM^2}{R^4}$$

So that using the temperature from the first and P_{gas} from the second,

$$\frac{P_{rad}}{P_{gas}} = \frac{\frac{1}{3}a\left(\frac{1}{k_B}\frac{GMm_p}{R}\right)^4}{\frac{3}{20\pi}\frac{GM^2}{R^4}} \approx 10^{-4}\left(\frac{M}{M_\odot}\right)^2$$

the radius cancels. (we'll come back to this combination of constants in a bit.) Note that $M > 100M_\odot$ that $P_r > P_g$ and there's trouble. The star cannot be held together stably.

Putting this back in gives

$$\frac{K_r}{K_G} = 10^{-4}\left(\frac{M}{M_\odot}\right)^2\left(\frac{m_ec^2}{kT}\right)^{5/2}$$

To have $K_r > K_G$, we just need

$$M > 0.03M_\odot\left(\frac{T}{10^7K}\right)^{5/4}$$

For most stars we will consider, γ's carry the heat from the deep interior.

8.1 Deriving stellar luminosity

Now that we know photons dominate, do they give the right luminosity?

Luminosity is set by the temperature gradient,

$$L = 4\pi R^2\frac{1}{3}c\frac{1}{n_e\sigma_\gamma}\frac{d}{dr}(aT^4)$$

for an estimate $kT = GMm_p/R$. Showing the scaling, dropping all constants

$$L \sim R^2\frac{R^3}{M}\frac{1}{R}\frac{M^4}{R^4} \sim M^3$$

radius cancels! When heat transport is via photons whose cross section is Thomson scattering:

$$L = L_\odot \left(\frac{M}{M_\odot}\right)^3$$

true for $M > M_\odot$ and is independent of radius.

This allow us to understand some basic differences between stars of different masses.

The nuclear evolution time is

$$t_{\rm nuc} \sim \frac{E_{\rm nuc}}{L}$$

Since the available energy in nuclear burning is just proportional to the mass (I used 7 MeV per m_p before). So then we find

$$t_{\rm nuc} \propto \frac{M}{L} \propto \frac{M}{M^3} \propto M^{-2}$$

Thus if the sun will live a few billion years, a $10M_\odot$ will live a few tens of millions of years. This is roughly correct.

With this we have a rough idea, based on the sun, of how the Luminosity and lifetime, and some aspects of the structure depend on mass. We used those structural relations to infer an interior $T \sim 10^7$ K from the observed size of the Sun.

8.2 Solar mass from fundamental constants

Now we want to go back and see how the characteristic mass scale of $100M_\odot$ for the maximum mass scale of a star, when the radiation pressure starts to dominate. Using the estimates we used last time

$$kT \sim \frac{GMm_p}{R} \quad \text{and} \quad \frac{GM^2}{R} \sim V\langle P\rangle \sim R^3 P_{gas}$$

we can write

$$\frac{P_{rad}}{P_{gas}} \simeq \frac{a\left(\frac{GMm_p}{k_B R}\right)^4}{GM^2/R^4} \simeq \frac{aG^3M^2m_p^4}{k_B^4}$$

so then with $a = \frac{\pi^2}{15}k_B^4/(\hbar c)^3$

$$\frac{P_{rad}}{P_{gas}} \simeq \frac{G^3m_p^4}{(\hbar c)^3}M^2 \simeq \frac{G^3m_p^6}{(\hbar c)^3}\left(\frac{M}{m_p}\right)^2$$

so then

$$\sim \left[\frac{Gm_p^2}{\hbar c}\right]^3 \left(\frac{M}{m_p}\right)^2$$

the constant out front is dimensionless so then if we define

$$\alpha_G = \frac{Gm_p^2}{\hbar c} = 6 \times 10^{-39}$$

with (in cgs) $G = 6.7 \times 10^{-8}$, $m_p = 1.7 \times 10^{-24}$, $\hbar c = 2000\text{eV Å}$, eV$= 1.6 \times 10^{-12}$. then

$$M = m_p \left(\frac{1}{\alpha_G} \right)^{3/2} = 2M_\odot \ .$$

when $P_{rad} \sim P_{gas}$.