6 Stars notes 2019/09/04 - Wed - Hydrostatics and Thermodynamics of stars

6.1 Self-gravitating objects

Objects where gravity is playing the important role for the structure. Want some basic understanding of the first two equations of stellar structure, before we get to heat transport.

First we want to justify using hydrostatic balance. What if the star was NOT in hydrostatic balance?

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} P + \rho \vec{g}$$

(no rotation or magnetic field). Then what if $P \to 0$? we see that

$$\frac{d\vec{v}}{dt} = \vec{g}$$

for a star,

$$\vec{g} = -\frac{Gm(r)}{r^2}\vec{\hat{r}}$$

where the mass interior to r is

$$m(r) = \int_0^r \rho(r) 4\pi r^2 dr$$

So then we want to know how long until collapse if we turn off the pressure then taking v_r pointing in

$$\frac{dv_r}{dt} = \frac{Gm(r)}{r^2}$$

Want to consider the **Dynamical time** also known as the free-fall time, the timescale on which a star would change in NOT in hydrostatic balance..

Note in spherical collapse there is no "shell crossing", i.e. the mass interior to a point is a good coordinate. Define v_r to be the velocity at the outer shell where $m(r) = m_0$ always. (the mass within a shell defined by a particle position is constant) then we have

$$\frac{d^2 r}{dt^2} = -\frac{Gm}{r^2}$$

which is the equation for collapse of a shell. Now we can integrate this equation to get $r(t)$, but what matters here is the characteristic timescale. Just grossly (finite-differencing):

$$\frac{r}{t_{Dyn}^2} \sim \frac{Gm}{r^2} \quad \Rightarrow \quad t_{Dyn} \simeq \frac{1}{\sqrt{Gm/r^3}} \sim \frac{1}{\sqrt{G\rho}}$$

For the sun

$$\langle \rho \rangle = \frac{M_\odot}{4\pi R_\odot^3/3} \simeq 1.4 gr/cc$$

and with $G \sim 10^{-7}$ in cgs units $t_{Dyn} \sim 1$ hour. the orbital period at the surface of an object is roughly the dynamical time. Kepler's law:

$$\omega_k^2 = \frac{GM}{r^3}$$

so then

$$\left(\frac{2\pi}{P_{orb}}\right)^2 = \frac{GM}{r^3}$$

so modulo constants these are the same.

Since the stars have been sitting there for a long time, we always presume hydrostatic balance.

6.2 Energetics of hydrostatic star

6.2.1 Virial theorem

hydrostatic assumption relates gravitational to thermal kinetic energy

Take

$$\frac{dP}{dr} = -\rho\frac{Gm(r)}{r^2}$$

multiply both side by $4\pi r^3 dr$ and integrate. Then RHS

$$RHS = -\int \rho(r)\frac{4\pi r^3 dr Gm(r)}{r^2}$$

but $dm(r) = \rho 4\pi r^2 dr$ so

$$= -\int \frac{Gm(r)dm(r)}{r} = E_G$$

but this is the gravitational binding energy:

$$E_G = -\int_0^M \left[\frac{Gm(r)}{r}\right] dm(r)$$

Now working on the other side

$$LHS = \int_0^M \frac{dP}{dr}4\pi r^3 dr$$

integrating by parts

$$= 4\pi r^3 P|_0^R - 3\cdot 4\pi\int P\cdot r^2 dr$$

for a star effectively $P(R) = 0$ since the central pressure is so much higher than the pressure from the ISM, and $r = 0$ at center,

$$LHS = 0 - 3\int P(r)4\pi r^2 dr$$

This looks much like the volume integral of kinetic energy, since P is a measure of the kinetic energy.

Define

$$\langle P\rangle = \frac{1}{V}\int P4\pi r^2 dr$$

so then we have from hydrostatic balance that

$$\boxed{\langle P\rangle = -\frac{1}{3}\frac{E_{GR}}{V}}$$

which is the virial theorem for the star. The total energy is

$$E_{tot} = E_G + E_{KE}$$

we know that $E_G = -3V\langle P\rangle$ so this says that

$$\boxed{E_{tot} = -3V\langle P\rangle + E_K} \quad \text{for any hydrostatic object}$$

so now we need some way to relate the pressure to the kinetic energy. So the relation between the pressure and the kinetic energy density of the the pressure-supporting particles really sets E_{tot}.

6.2.2 Total binding energy and particle type

Relation between pressure and KE density Take a box with particles in it which have three degrees of freedom. These have some number density $n = N/V$. Imagine that $1/6$ of them are each hitting each wall at the same time. So the number per second striking the wall

$$r = A\frac{n}{6}v$$

(this doesn't give you the $1/3$ honestly, but it shows the difference between relativistic and non-relativistic) the momentum transferred into the wall by each particle is $\Delta p = 2p$, where p is the momentum of a particle, so then the force on the wall is $F = 2p\frac{An}{6}v$ so the pressure is

$$\frac{F}{A} = P = \frac{1}{3}pnv\ .$$

So far we haven't put in relativistic or non-rel.

Non-relativistic Start with non-relativistic, so that $p = mv$ and then

$$P = \frac{1}{3}(mv)nv = \frac{2}{3}n\frac{1}{2}mv^2 = \frac{2}{3}n(e_{KE})$$

where we have used e_{KE} for the "kinetic" internal energy of a gas particle. i.e. the thermal energy per particle. So the term in the virial theorem is

$$-3V\langle P\rangle = -3V\frac{2}{3}n(e_{KE}) = -2Ne_{KE} = -2E_{KE}$$

So the virial theorem itself,

$$-3V\langle P\rangle = E_G \quad \text{becomes} \quad -2E_{KE} = E_G \quad \text{or} \quad \boxed{E_{KE} = -\frac{1}{2}E_G}$$

And so for non-relativistic, the total binding energy is:

$$E_{tot} = \quad -3V\langle P\rangle + E_{KE} = -2E_{KE} + E_{KE} \quad \boxed{E_{tot} = -E_{KE} = \frac{1}{2}E_G}$$

.

Relativistic What if particles are relativistic?:
$p = E/c = e_{KE}/c$ and $v = c$

$$P = \frac{1}{3}pnv = \frac{1}{3}\frac{e_{KE}}{c}nc = \frac{1}{3}n(e_{KE}) = \frac{1}{3}\frac{E_{KE}}{V} \ .$$

So if relativistic

$$E_{tot} = -3V\langle P\rangle + E_{KE} = 0 \quad (!!!)$$

So if a hydrostatic object is supported by relativistic particles, its binding energy is zero and independent of things like its size and temperature. Can't really make a stable object that way.

To say where each occurs: relativistic particles supply the pressure in two places

1. Massive ($M > 50M_\odot$) stars, supporting particles are photons.
2. Massive White dwarfs, supported by relativistic electrons, leads to the Chandrasekhar mass.

but for now we'll focus on the non-relativistic case.

6.3 Thermodynamics of star

In stars, we have just found that

$$E_{tot} = -K_{KE} = \frac{1}{2}E_{Gr}$$

so that as a star loses overall energy, the kinetic energy of the particles increases, and thus T increases. So the heat capacity of a star is negative. To put it the other way, putting energy into a star and causes the temperature to go down. True for self-gravitating and in hydrostatic balance, because the thermal and gravitational energy resivoirs are coupled.

MESA exercise:
Try it! compute the total energy as the sum of the gravitational potential energy (formula shown above) and the integral of the thermal internal energy. (you can actually get mesa to include these in the history output file) Plot this and the central temperature vs. time during the time between formation and hydrogen ignition in the getting started simulation. You could also plot the luminosity and integrate it to see where all that energy goes!