5 Stars notes 2019/09/01 - Fri - Hydrostatic balance

5.1 Hydrostatic Balance continued

Will discuss hydrostatic balance in plane-parallel today, and in full star next time.

Isothermal, plane-parallel atmosphere gravity pointing down, z pointing up. Hydrostatic balance

$$\frac{dP}{dz} = -\rho g$$

and with

$$P = \frac{\rho kT}{\mu m_p}$$

(in cgs $1/m_p$ is a mol!) recall that

$$\frac{1}{\mu} = \frac{1}{\mu_{ion}} + \frac{1}{\mu_{ele}} = \sum_i \frac{X_i}{A_i} + \sum_i \frac{Z_i X_i}{A_i}$$

then

$$\frac{kT}{\mu m_p} \frac{d\rho}{dz} = -\rho g \quad \text{or} \quad \frac{d\rho}{dz} = -\frac{\rho}{H}$$

where $H = kT/\mu m_p g$ is called the Scale Height. Also $H = P/\rho g$. very imortant. this is the characteristic length scale for an atmosphere. so we obtain

$$\rho(z) = \rho(0) \exp\left(\frac{-z}{H}\right) \quad \propto \quad \exp\left(\frac{-\mu m_p g z}{kT}\right) = \exp\left(-\frac{E}{kT}\right)$$

Looks like a boltzmann factor, giving the probability of finding a particle at a particular height. The scale height is the distance that the particle needs to fall to release kT in energy.

Now we can ask some questions. such as how thick is the atmosphere compared to the radius of the object?:

$$\frac{H}{R} = \frac{kT/\mu m_p g}{R} = \frac{kT}{\mu m_p} \frac{R^2}{GMR} = \frac{kTR}{\mu m_p GM} = \frac{kT}{GM\mu m_p/R}$$

this just asks what is the ratio of the thermal energy to the gravitational binding energy. This is why stars have sharp edges – their surface temperatures are small compared to their binding energy. For example, for the sun this is like a part in 1000, $kT \sim$eV (hydrogen ionization) while $GM\mu m_p/R \sim$keV. similar for the earth.

MESA exercise:

Plot from structure profile of ZAMS Sun: Pressure vs. radius.

What is the length over which the pressure increases by a factor of unity (i.e. pressure doubles). What about density? How does this compare to the above H/R estimate? (using the R of the structure you have)

Column density Often the heat transport is closely related to the "column density" so we like this quantity. What we want to know is what is the amount of stuff above particular point integrating from infinity. (this sets how hard it is for heat to escape from this point).

$$y(z) = \int_z^\infty \rho(z)dz = gram/cm^2$$

about 1000 g/cm^2 for earth's atmosphere at surface. Want to relate this to the pressure. In hydrostatic balance

$$\int_z^\infty dP = -\int_z^\infty \rho(z)gdz$$

or

$$P(\infty) - P(z) = -g\int_z^\infty \rho(z)dz$$

so then

$$P(z) = g\int_z^\infty \rho(z)dz = gy(z)$$

So what is the column for our isothermal atmosphere?

$$y(z_0) = \int_{z_0}^\infty \rho(z_0)e^{-(z-z_0)/H}dz = \rho(z_0)H$$

nice and simple, even geometric.

5.2 Electric fields in ionized atmospheres

But stellar matter is not an ideal gas - it is a plasma - basically a negatively and positively charged gas occupying the same space. How does this work out?

Imagine a purely ionized H atmosphere. so then $n_e = n_p$ (by definition of charge balance.) Then

$$P = (n_e + n_p)kT = 2n_pkT$$

Hydrostatic balance gives that $\frac{dP}{dz} = -\rho g$ and then for a isothermal atmosphere, (T=constant with z)

$$n_p \propto e^{-z/H} \quad H = \frac{2kT}{m_pg}$$

i.e. $\mu = 1/2$.

The question is why don't the protons sink? Write down hydrostatic balance for each species, e+p. assume an electric field up and gravity down.

$$\frac{1}{n_e}\frac{dP_e}{dz} = -m_eg - eE$$

and for protons

$$\frac{1}{n_p}\frac{dP_p}{dz} = -m_pg + eE$$

adding these gives hydrostatic balance. taking their difference (with $n_e = n_p$, so that $P_e = P_p$ also) gives

$$0 = -m_e g + m_p g - 2eE \qquad eE = \frac{1}{2} m_p g$$

So the electric field acts to lighten the proton by a factor of two and is what holds on to the electron. So the macroscopic forces on the electron and the proton are identical so there is no relative sedimentation. in the plane parallel case this is a field just like capacitor plates.

Generally only necessary for diffusion of species in the star.