25 Astro notes 2018/10/22 - Mon - GR: intervals, Schwarzchild metric

25.1 Proper intervals

As we have seen in special relativity, the "distance", if defined as the same-time separation, between separated points can depend on the speed of the observer. i.e. it is not a universal thing. The analog of distance that is not observer-dependent is the spacetime interval.

$$I = (\Delta s)^2 = [c(t_2 - t_1)]^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

(this assumes a flat spacetime) note there is a sign convention ambiguity here, so that some prefer

$$I = -[c(t_2 - t_1)]^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Whether the interval I is positive or negative determines if it is timelike or spacelike, based on adopted sign convention. Note that in flat space without motion (or without time elapsed), the interval reduces to either the time (or space) separation. "proper time" and "proper distance" are defined this way, in a frame at rest with respect to the object/clock under consideration.

Note that:

- Particle paths are exclusively timelike.
- Events with spacelike separations cannot be causally connected.
- Light moves on zero interval paths.

But we have said that space is local, not global like these intervals imply. This leads us to instead define the local "metric" for how to measure small pieces of intervals:

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

Now intervals are found by integrating along a worldline

$$\Delta s = \int_{path} \sqrt{(ds)^2}$$

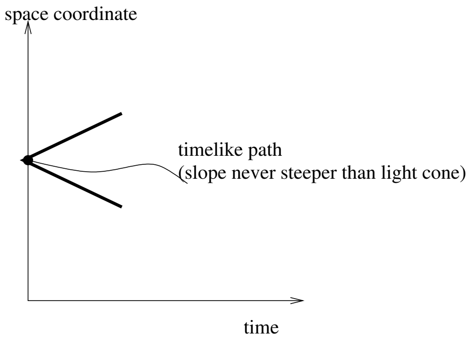
We can now actually finally compute the time interval experienced by a clock moving on an arbitrary path:

$$\Delta t_{proper} = \frac{\Delta s}{c}$$

This works for any path in any coordinate system and does not depend on anything special about the "observer". Computing the time interval elapsed for an observer in flat space is a similar computation, but with a path that has no spatial components (at rest), and therefore matches the "t" coordinate.

If we want to find the path of a free object, that is a bit more complex. A free-falling object follows a geodesic – a path that extremizes the spacetime interval (max or min depending on sign convention). For flat space that is a straight line path.

25.2 Clarity on sign convention



With the + - - - convention, as in the text, positive spacetime intervals are *timelike*. So the proper time (the time measured by a clock moving along a given path) is given by

$$\Delta\tau = \frac{\Delta s}{c} = \frac{1}{c}\int_{path}\sqrt{(ds)^2}$$

where ds^2 is given from the coordinates by the metric. Note that if ds^2 is negative (spacelike) *at any point in the path* the proper time is not defined. For purely spacelike paths the spacelike distance is

$$L = \int_{path}\sqrt{-(ds)^2}$$

In this case also free particle paths are geodesics with maxiumum spacetime interval.

With the - + + + convention,

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

and negative spacetime intervals are timelike. So the proper time is

$$\Delta\tau = \frac{\Delta s}{c} = \frac{1}{c}\int_{path}\sqrt{-(ds)^2}$$

Now if ds^2 is positive (spacelike) the proper time is again not defined. In this case free particle paths are geodesics with minimum spacetime interval (i.e. most negative spacetime interval).

25.3 Curved space - the Schwarzschild metric

I will give one example of curved space - the metric for the space around a spherically symmetric gravitating object like a planet or star or black hole. This is

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 - \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

Note, for example, that this has gravitational time dilation built into it. The time interval for a clock at some radius r is different than for one further out.

For comparison, the metric of flat space-time (i.e. no gravitational sources) in similar coordinates is

$$ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

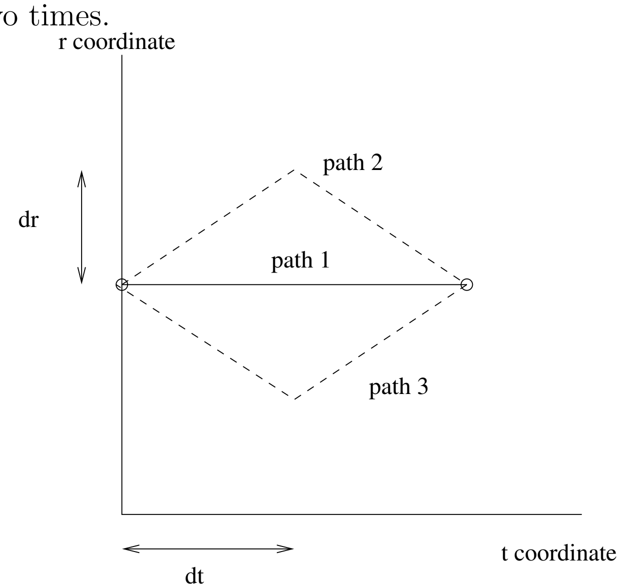
We see that, for the Schwarzschild metric, as represented by the rubber sheet analogy, the distance between r and $r + dr$ in space-time is

$$\Delta s = \frac{dr}{\sqrt{1 - 2GM/rc^2}}$$

which is larger than dr for all radii.

But if we integrate a circle at a fixed r in either metric (fixing $\theta = \pi/2$ and running ϕ from 0 to 2π so $ds = rd\phi$) we obtain the circumference $2\pi r$.

We said that particles move along paths of extremal distance. We will investigate this by first showing that in the flat spacetime metric (also called Minkowski spacetime) a straight path is the longest spacetime interval. Consider the following paths through spacetime (we will only use the radial space coordinate). So this is a free-falling path with the same r at two times.



Now we will compare the spacetime interval Δs integrated along each paths. For flat space, path 1 is

$$\Delta s_1 = 2cdt$$

For path 2 we have

$$\Delta s_2 = 2\sqrt{c^2dt^2 - dr^2}$$

And since the sign of dr doesn't matter due to the square, this is also the length of path 3. So we see that for any $dr > 0$, that path 1 has a larger spacetime interval than path 2 or 3. i.e. it is the extremal path and therefore the one along which a particle would travel between these the initial and final points. Also we find from here that the extremum we want for a particle path is actually the maximum.

next time we will consider the Schwarzchild metric...