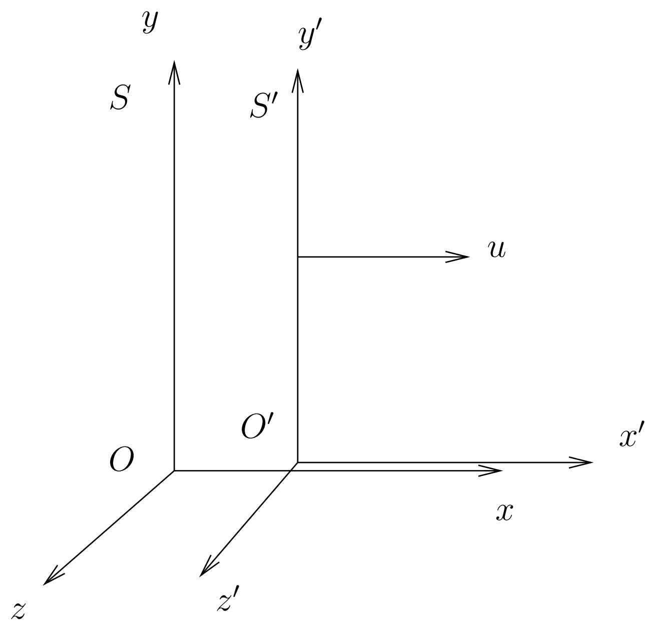
7 Astro notes 2018/9/7 - Fri - special relativity

7.1 Intro Special Relativity

Galilean transformation stems from assuming that it makes sense to speak of the global structure of space and time. i.e. there are coordinates where it is well defined to measure the position, x, y, z, and time t of *events*. One can then imagine two such coordinate systems that are in relative motion at constant velocity. We call the second coordinate system the "prime" system, which is moving at speed u in the x direction.



Then we can easily convert events (positition, time) sets from the base coordinates to the prime coordinates, assuming that O and O' are coincident at $t = 0$. That is $x = x'$ at $t = t' = 0$.

Galilean:

$$x' = x - ut$$

$$y' = y, \quad z' = z, \quad t' = t$$

Then if we have a particle path $\vec{x}(t)$, its velocity $\vec{v}(t)$ can be computed in prime coordinates, giving

$$v'_x = v_x - u, \quad v'_y = v_y, \quad v'_z = v_z$$

Newtons laws, phrased in terms of $d^2\vec{x}/dt^2$ are then unchanged.

Problems arise when one considers that there is some reference frame in which the speed of light is the value obtained from Maxwell's equations. This is equivilent to there being some correct definition of electric vs. magnetic field. Recall that static charges generate electric fields, and moving charges generate magnetic fields, so which field is present appears to depend on how the chosen reference frame moves.

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \qquad \vec{\nabla} \times \vec{B} = \frac{1}{c}\left(4\pi\vec{J} + \frac{\partial \vec{E}}{\partial t}\right)$$

(not only does $\vec{J}$ depend on the speed of the frame, but so does the time dependence of ρ, which enters via $\partial\vec{E}/\partial t$.) This seems, at first glance to imply that there is some favored reference frame, though it turns out that it implies exactly the opposite!

This was first shown experimentally by attempting to measure the speed of light in opposite directions. If there is some favored reference frame, it should then be easy to find ones relative motion with respect to it. It turned out, despite trying, no difference in the speed of light was ever measured.

7.2 Lorentz transformations

It turns out the concept of a well-defined global coordinate system is what is flawed. Local connections between events based on the propagation speed of light, however, does provide a way to construct a workable concept of spacetime. Einstein expressed this in two postulates:

- The laws of physics are the same in all inertial reference frames.
- Light moves through vacuum at a constant speed c independent of source.

These two basically tell you how to construct valid reference frames for Newton's laws (inertial) and how they are connected (all measure the same speed for light).

Consider the general coodinate transformation:

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

etc. This is a fully general linear transformation. A transformation of similar form, but corresponding to a different operation, is that of a rotated reference frame:

$$x' = (\cos\theta)x - (\sin\theta)y \qquad y' = (\sin\theta)x + (\cos\theta)y$$

The Galilean transform is also of this form.

We seek a transform between frames where the prime frame is moving at constant speed in the x direction as before. The transverse directions are nominally unchanged, so that

$$y' = y, \quad z' = z$$

The symmetry with respect to y or z also implies that the time transformation should not depend on them. So

$$t' = a_{41}x + a_{44}t$$

Finally for the x coordinates, we have defined the relative motion of the coordinate system origins. They coincide at $t = 0$ and the origin of the prime coordinate system ($x' = 0$)

is moving with speed u. Thus the coordinates of the origin of the prime coordinates are $x' = 0$ and $x = ut$.

(Student:) What does this tell us?

$$0 = a_{11}ut + a_{12}y + a_{13}z + a_{14}t$$

since this holds on the yz plan for any y, z, $a_{12} = a_{13} = 0$. So we are left with $a_{11}u = -a_{14}$. So far we have

$$x' = a_{11}(x - ut), \qquad t' = a_{41}x + a_{44}t$$

To set the final coefficients, consider the propagation of a light pulse started at the origin at time zero. Invoking the second of Einstein's postulates this gives the location in both frames

(Student:) How is this written in each coordinate system?

$$x^2 + y^2 + z^2 = c^2t^2, \qquad x'^2 + y'^2 + z'^2 = c^2t'^2$$

Putting in the above we get

$$a_{11}^2(x^2 - 2xut + u^2t^2) + y^2 + z^2 = c^2(a_{41}^2x^2 + 2a_{41}a_{44}xt + a_{44}^2t^2)$$

or

$$(a_{11}^2 - c^2a_{41}^2)x^2 + y^2 + z^2 = (a_{11}^2u + c^2a_{41}a_{44})2xt + (c^2a_{44}^2 - a_{11}^2u^2)t^2$$

Matching up, the t^2 term tells us

$$c^2a_{44}^2 - a_{11}^2u^2 = c^2$$

$$a_{11}^2 - c^2a_{41}^2 = 1$$

$$a_{11}^2u + c^2a_{41}a_{44} = 0$$

or

$$a_{11}^4u^2 = c^2a_{41}^2c^2a_{44}^2 = (a_{11}^2 - 1)(c^2 + a_{11}^2u^2) = a_{11}^2c^2 - c^2 + a_{11}^4u^2 - a_{11}^2u^2$$

so that

$$a_{11}^2 = \frac{1}{1 - u^2/c^2}$$

and then

$$a_{44} = a_{11} \qquad a_{41} = -ua_{11}/c^2$$

so that our transformations are

$$x' = \gamma(x - ut), \qquad t' = \gamma(t - ux/c^2), \qquad y' = y, z' = z$$

where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

is called the Lorentz factor.

7.3 Summary

we obtained the Lorentz transformation, the relation between the coordinates of events in a reference frame and one moving at speed u with respect to it in the x direction.

$$t' = \gamma(t - ux/c^2)$$

$$x' = \gamma(-ut + x)$$

$$y' = y, \qquad z' = z$$

where $\gamma = 1/\sqrt{1 - u^2/c^2}$, and note that $\gamma \geq 1$ always.

We will also make use of the inverse transforms which are most easily obtained by swapping u to $-u$ and the primes to non-primes

$$t = \gamma(t' + ux'/c^2)$$

$$x = \gamma(ut' + x')$$

7.4 Simultaneity

Using these transformations we can compute the time, in the moving frame, between two events which are simultaneous (equal t) in the base frame.

$$t'_2 - t'_1 = -\frac{(x_2 - x_1)u/c^2}{\sqrt{1 - u^2/c^2}}$$

thus the events are not simultaneous in the prime frame. Also which one is first actually depends on relative sign of u compared to $x_2 - x_1$, i.e. the direction of the motion.

7.5 Homework problem on simultaneity (or, rather, lack thereof)

Handed out:

Consider a spaceship that is traveling past the solar system toward a star 10 light-years away at $0.5c$. A signal is emitted from Earth when the spaceship is halfway there, and therefore received just as the spaceship reaches the star.

Draw, to scale as much as possible, a spacetime diagram in both the solar system rest frame and a frame moving with the spaceship that shows the path of the Sun, the distant star, the spaceship and the signal. How far does the signal travel in the frame of the spaceship? By working back from the time the signal was received using the speed of light and the varying distance to the solar system, what fraction of the distance to the star would the occupants of the spaceship say that they had covered at the time, according to their coordinates, that the signal was emitted? (That is, what is the time coordinate of the emission event in the spaceship's frame, and how does this compare to the time coordinate of the spaceship arriving at the star?)

You should explicitly give the coordinates in both the solar system and spaceship frames of the followin events:

- Departure of spaceship from solar system
- Arrival of spaceship at star
- Emission of signal
- Arrival of signal at star

You must also compute and show the positions of both the solar system and the distant star at the beginning and end of the trip in both frames.