5 Astro notes 2018/08/31 - Fri - Light 3: Thermal spectrum, Colors

Last time we introduced the flux density, which is a function F such that energy can be added up using

$$dE = F_\lambda(\lambda, \hat{n}, \vec{x}) dA d\lambda d\Omega dt$$

As a result, F has dimensions of energy per time per area per wavelength per solid angle. Usually the units are ergs/ (s-cm^2-Å-steradian).

The simplist example is for a uniform background (no anglular dependence), with a pixel subtending a solid angle Ω the flux collected is $f_\lambda = F_\lambda \Omega$.

For a star, the full flux density looks like

$$F_\lambda = \frac{B_\lambda(T, \lambda) R^2}{|\vec{x} - \vec{x}_s|^2} \delta(\hat{n} - \widehat{\vec{x} - \vec{x}_s})$$

So that if the angles subtended by the pixel include the direction toward the star, we recover the previous expression for flux $f = L/4\pi d^2$. But we really need a bit more info about the blackbody function...

5.1 Thermalized radiation (Blackbody)

Without any other assumptions, there is no reason to necessarily believe that the wavelengith dependence of flux, $F(\lambda)$, would be simple. Independent wavelengths can propagate indepently. However it is possible to reduce the wavelength dependence to a single parameter, T, the temperature if we assume **thermal equilibrium** in the radiation field. This can be posed in several ways, the most common being a box with a small hole to the outside, where the photon field has ample opportunity to equilibriate inside the box. A fully ionized, optically thick medium, such as the interior of a star, also has a photon field which is thermalized, and therefore follows the same distribution.

The construction of the Plank function comes from assuming that creating light occurs by creating photons whose energy is related to their wavelength,

$$E_\gamma = h\nu = \frac{hc}{\lambda}$$

where h is Planck's constant. It turns out that this is a deep meaningful statement about the nature of radiation that *does not follow from Maxwell's equations.* By doing this, it is possible to determine statistically what the **occuption of different wavelengths** should be in thermal equilibrium from which Planck's function is obtained:

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1}$$

where k is Boltzmann's constant. Note this is energy/s per wavelength interval per area per steradian. So flux in power is $\int B_\lambda(T)d\lambda$. This means that if we want the frequency-specific power, we must convert to $B_\nu(T)d\nu$, so that

$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1} \ .$$

Fluxes must also be given either per wavelenth or per frequency, thus you will see F_λ and F_ν to denote the difference.

There are two interesting limits to take. The first is long wavelength. In this case we can expand $e^{hc/\lambda kT} \simeq 1 + hc/\lambda kT$. This gives

$$B_\lambda(T) \simeq \frac{2hc^2}{\lambda^5}\frac{\lambda kT}{hc} = \frac{2ckT}{\lambda^4}$$

which is the **Rayleigh-Jeans Law**.

Also if λ small, or, equivalently, ν is large, we find

$$B_\nu(T) \simeq \frac{2h\nu^3}{c^2}e^{-h\nu/kT}$$

which follows naturally from classical thermal physics.

Wein's Law Finally it is straight forward to derive the maximum of $B_\lambda(T)$ in wavelength just by taking derivatives

$$\frac{dB_\lambda}{d\lambda} = 0 = B_\lambda\frac{-5}{\lambda} + \frac{B_\lambda}{e^{hc/\lambda kT} - 1}e^{hc/\lambda kT}\frac{hc}{\lambda^2 kT}$$

or

$$\frac{hc}{\lambda kT}\frac{e^{hc/\lambda kT}}{e^{hc/\lambda kT} - 1} - 5 = 0$$

The solution to this equation is $hc/\lambda kT = 4.97...$ giving

$$\lambda_{max}T = 0.002898 \text{ m K}$$

5.2 Effective temperature

The thermal flux law can be used to obtain the flux from the surface of a star. The flux at the surface of a blackbody is

$$F_\lambda = B_\lambda \cos\theta$$

where θ is the angle from the normal. At every point on the surface of the star we must integrate over the upper half-volume above the surface

$$F_{surf} = \int_{\lambda=0}^{\infty}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} B_\lambda \cos\theta\, \sin\theta\, d\theta\, d\phi = \sigma_{\rm SB}T^4$$

where σ_{SB} is the Stefan-Bolztmann constant. This is often generalized so that even if F_λ is not a blackbody, an "effective" temperature can be defined such that $F_{\rm surf} = \sigma_{\rm SB} T_{\rm eff}^4$.

But so far we have only integrated F over direction (with respect to the surface) and wavelength. We still must integrate over the surface to find the total energy release:

$$L = \int_{surf} \sigma_{\rm SB} T^4 = 4\pi R^2 \sigma_{\rm SB} T^4$$

where R is the stellar radius.